Linear functions

Some of the most important functions are linear. This unit describes how to recognize a linear function, and how to find the slope and the y-intercept of its graph.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- recognise when a rule describes a linear function,

- be able to plot the graph of part of a linear function,

- find the slope and the y-intercept of the graph of a linear function.

Contents

1. Linear functions

A linear function is a function of the form $f(x) = ax + b$, where a and b are real numbers. Here, a represents the gradient of the line, and b represents the y-axis intercept (which is sometimes called the vertical intercept).

What do you think will happen if we fix b and vary a? Let us try some examples. Since the functions we are looking at are linear, the graph will be a straight line. So we only need two points to be able to draw the line. However, we generally choose three, and the third point is a good check that we haven't made a mistake.

(a) $f(x) = x + 2$: $\quad f(0) = 2, \quad f(1) = 3, \quad f(2) = 4$
(b) $f(x) = 2x + 2$: $\quad f(0) = 2, \quad f(1) = 4, \quad f(2) = 6$
(c) $f(x) = -2x + 2$: $\quad f(0) = 2, \quad f(1) = 0, \quad f(2) = -2$
(d) $f(x) = -x + 2$: $\quad f(0) = 2, \quad f(1) = 1, \quad f(2) = 0$
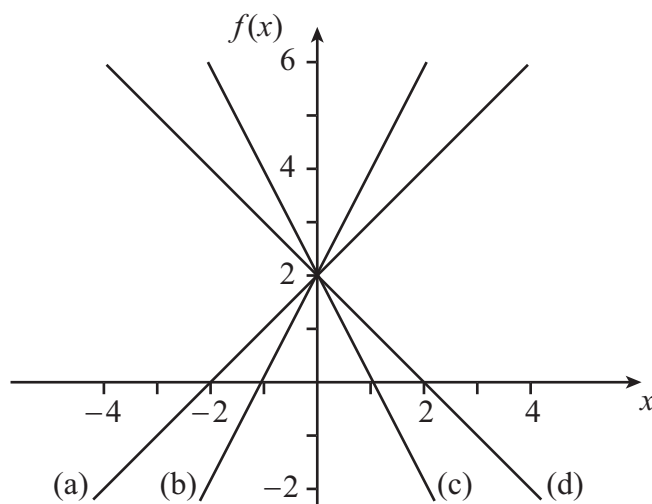
You can clearly see from our diagram that the graphs of all of the functions cross the y-axis at $y = 2$. This is because b is fixed as 2, and b represents the y-axis intercept. You can see that if $a > 0$ then the straight line goes up as x increases, and the bigger a is the faster the line goes up. Similarly, if $a < 0$ then the line goes down as x increases, and the bigger a is in absolute terms, the faster the line goes down.

What do you think will happen if we fix a and vary b? Again we can try some examples.

$$
\begin{array}{llll}
\text{(a)} & f(x) = 2x + 3 : & f(0) = 3, & f(1) = 5, & f(2) = 7 \\
\text{(b)} & f(x) = 2x + 1 : & f(0) = 1, & f(1) = 3, & f(2) = 5 \\
\text{(c)} & f(x) = 2x - 3 : & f(0) = -3, & f(1) = -1, & f(2) = 1
\end{array}
$$
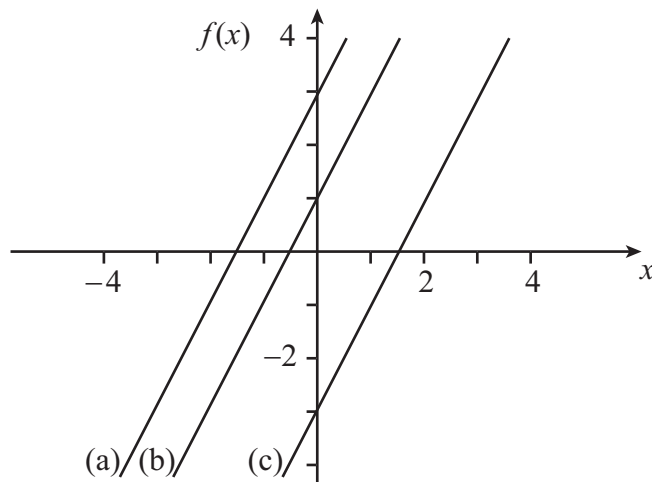
When you look at the graphs of the functions, you can see straight away that they all have the same gradient. This is because a is fixed as 2, and represents the gradient. You should also notice that b represents the y-axis intercept (that is, the vertical intercept) in each case.

Now we know what happens when a and b are positive or negative, but what happens if either of them is zero? Suppose that $a = 0$. Then we would have functions of the form $f(x) = b$ where b is constant, for example $f(x) = 2$ or $f(x) = -3$.
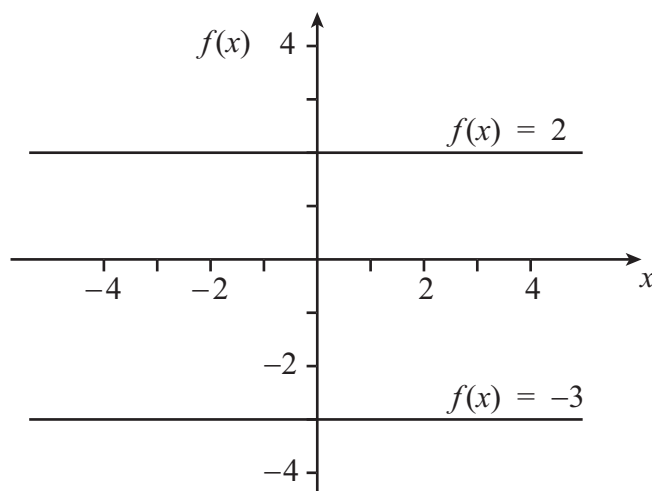
You can see that a gradient of zero always gives a horizontal line, and that the line cuts the y-axis at b.

Suppose instead that b = 0. Then we would have functions of the form $f(x) = ax$, for example $f(x) = 2x$ or $f(x) = -3x$.
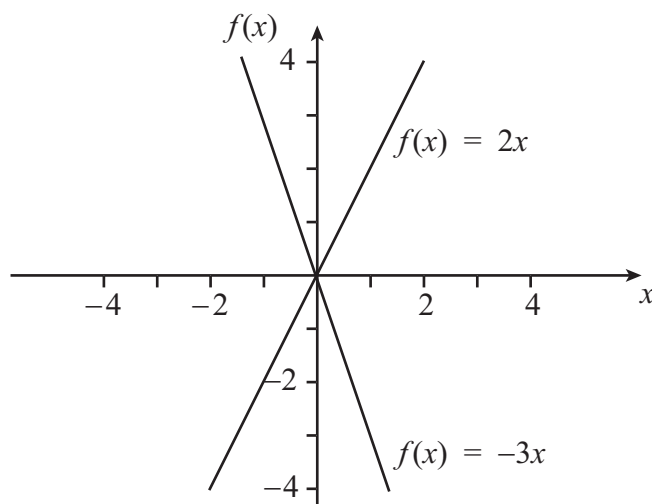
The y-axis intercept would be equal to zero, and so the graphs of all these functions pass through the origin, and the gradient of the line depends upon a.

2. Linear functions not written in standard form

Finally, let us consider what happens when we have linear functions that are not in the form $f(x) = ax + b$. For simplicity, we shall put $y = f(x)$ here.

Suppose that we have the equation $4x - 3y = 2$. To get this into the required form we need to make y the subject:

$$\begin{aligned} 4x &= 2 + 3y \\ 4x - 2 &= 3y \\ \tfrac{4}{3}x - \tfrac{2}{3} &= y \,. \end{aligned}$$

Then, since $y = f(x)$, this equation represents the function $f(x) = \tfrac{4}{3}x - \tfrac{2}{3}$. The graph of the function will have a gradient of $\tfrac{4}{3}$ and a y-axis intercept of $-\tfrac{2}{3}$.

Suppose that we have the equation $2x + 8y - 1 = 0$. To get this into the required form we need to make y the subject:

$$\begin{aligned} 2x + 8y &= 1 \\ 8y &= 1 - 2x \\ y &= \tfrac{1}{8} - \tfrac{1}{4}x \\ &= -\tfrac{1}{4}x + \tfrac{1}{8} \,. \end{aligned}$$

Then, since $y = f(x)$, this equation represents the function $f(x) = -\tfrac{1}{4}x + \tfrac{1}{8}$. The graph of the function will have a gradient of $-\tfrac{1}{4}$ and a y-axis intercept of $\tfrac{1}{8}$.

Suppose that we have the equation $y = (13x - 8)/5$. Although y is already the subject, this is still not in the required form. We need to deal with the division by 5: so $y = \tfrac{13}{5}x - \tfrac{8}{5}$. Then, since $y = f(x)$, this equation represents the function $f(x) = \tfrac{13}{5}x - \tfrac{8}{5}$. The graph of the function will have a gradient of $\tfrac{13}{5}$ and a y-axis intercept of $-\tfrac{8}{5}$.

Key Point

Functions of the form

$$f(x) = ax + b$$

are linear, and they are represented graphically by straight lines. The number a represents the gradient of the line, and the number b represents the y-axis intercept.

Exercises

1. What is a linear function?

2. By drawing up a table of values, plot the following linear functions on the same axes:

(a) $f(x) = 2x + 1$ (b) $f(x) = 3x - 2$ (c) $f(x) = 4 - 3x$ (d) $f(x) = 2 - x$

3. Find the gradient and the vertical intercept for each of the following linear functions by re-arranging them into the form $f(x) = ax + b$ (note: $y = f(x)$).

(a) $2y + 4x = 12$ (b) $5x - y = 9$ (c) $-3x = 1 - 4y$ (d) $2 - y/3 = x$
(e) $3 = 3y/4 - 2x/3$ (f) $12x - 4 = y/3 + 3$

4. Write down three different functions in which all the graphs are represented by parallel lines.

5. Write down three different functions in which all the graphs have the same vertical intercept.

Answers

1. A linear function is a function that can be expressed in the form $f(x) = ax + b$, where a and b represent real numbers.

2.

x	-2	-1	0	1	2
$2x + 1$	-3	-1	1	3	5
$3x - 2$	-8	-5	-2	1	4
$4 - 3x$	10	7	4	1	-2
$2 - x$	4	3	2	1	0
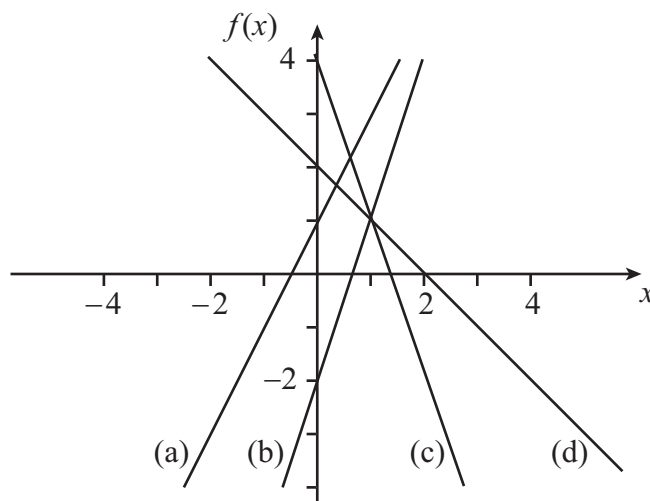
3.

(a)

$$
\begin{aligned}
2y + 4x &= 12 \\
2y &= 12 - 4x \\
y &= 6 - 2x \\
y &= -2x + 6 \\
f(x) &= -2x + 6
\end{aligned}
$$

Gradient $= -2$, vertical intercept $= 6$.

(b)

$$
\begin{aligned}
5x - y &= 9 \\
5x &= 9 + y \\
5x - 9 &= y \\
y &= 5x - 9 \\
f(x) &= 5x - 9
\end{aligned}
$$

Gradient $= 5$, vertical intercept $= -9$.

(c)

$$
\begin{aligned}
-3x &= 1 - 4y \\
4y - 3x &= 1 \\
4y &= 3x + 1 \\
y &= \tfrac{3}{4}x + \tfrac{1}{4} \\
f(x) &= \tfrac{3}{4}x + \tfrac{1}{4}
\end{aligned}
$$

Gradient $= \frac{3}{4}$, vertical intercept $= \frac{1}{4}$.

(d)

$$
\begin{aligned}
2 - \frac{y}{3} &= x \\
2 &= x + \frac{y}{3} \\
2 - x &= \frac{y}{3} \\
\frac{y}{3} &= 2 - x \\
y &= 6 - 3x \\
f(x) &= -3x + 6
\end{aligned}
$$

Gradient $= -3$, vertical intercept $= 6$.

(e)

$$\begin{aligned}
3 &= \frac{3y}{4} - \frac{2x}{3} \\
3 + \frac{2x}{3} &= \frac{3y}{4} \\
12 + \frac{8x}{3} &= 3y \\
4 + \frac{8x}{9} &= y \\
y &= \frac{8x}{9} + 4 \\
f(x) &= \frac{8x}{9} + 4
\end{aligned}$$

Gradient $= \frac{8}{9}$, vertical intercept $= 4$.

(f)

$$\begin{aligned}
12x - 4 &= \frac{y}{3} + 3 \\
12x - 7 &= \frac{y}{3} \\
36x - 21 &= y \\
f(x) &= 36x - 21
\end{aligned}$$

Gradient $= 36$, vertical intercept $= -21$.

4. Parallel lines have the same gradient, so we may take any three functions where a has the same value, for example

$$f(x) = 2x - 1, \qquad f(x) = 2x + 5, \qquad f(x) = 2x - \tfrac{1}{2}.$$

5. We may take any three functions where b has the same value, for example

$$f(x) = 4x - 5, \qquad f(x) = 2x - 5, \qquad f(x) = 25x - 5.$$